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SOME INTEGRAL FORMULAS AND THEIR APPLICATIONS TO HYPERSURFACES OF $S^n \times S^n$

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In his recent paper [4], Simons has established a fundamental formula for the Laplacian of the length of the second fundamental tensor of a submanifold of a Riemannian manifold and has obtained an application in the case of a minimal hypersurface of a sphere. Nomizu and Smyth [2] then obtained an important application of the formula of Simons' type to a hypersurface of constant mean curvature immersed in a space of nonnegative constant curvature.

On the other hand, Chern-do Carmo-Kobayashi [1] have obtained a classification theorem for submanifolds with the second fundamental tensor of constant length which is immersed in a sphere.

In this paper we discuss the same type of problem for compact orientable hypersurfaces with constant mean curvature immersed in $S^n \times S^n$.

In § 1 we review some fundamental formulas for a hypersurface of $S^n \times S^n$.

In § 2, using the formulas obtained in § 1 we establish an integral formula of Simons' type and obtain a theorem corresponding to that of Simons' paper.

In § 3 we consider an invariant hypersurface of $S^n \times S^n$ and prove some classification theorems corresponding to those of Chern-do Carmo-Kobayashi and of Nomizu-Smyth.

1. Hypersurfaces of $S^n \times S^n$

Let S^n be an n-dimensional sphere of radius 1, and consider $S^n \times S^n$. We denote by $\bar{P}$ and $\bar{Q}$ the projection mappings of the tangent space of $S^n \times S^n$ to each component S^n respectively. Then we have

$$\bar{P} + \bar{Q} = 1 , \tag{1.1}$$

$$\bar{P}^2 = \bar{P} , \qquad \bar{Q}^2 = \bar{Q} , \tag{1.2}$$

$$\bar{P}\bar{Q} = \bar{Q}\bar{P} = 0 . \tag{1.3}$$

We put

$$\bar{J} = \bar{P} - \bar{Q} . \tag{1.4}$$

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Then by virtue of (1.1), (1.2) and (1.3), we can easily see that

(1.5) $$\bar{J}^2 = I ,$$

(1.6) $$\operatorname{tr} \bar{J} = 0 ,$$

where $\operatorname{tr} \bar{J}$ denotes the trace of $\bar{J}$. We call $\bar{J}$ an *almost product structure* on $S^n \times S^n$.

We define a Riemannian metric on $S^n \times S^n$ by

$$\bar{g}(\bar{X}, \bar{Y}) = g'(\bar{P}\bar{X}, \bar{P}\bar{Y}) + g'(\bar{Q}\bar{X}, \bar{Q}\bar{Y}) ,$$

where g' is the Riemannian metric of S^n. Then it follows that

(1.7) $$\bar{g}(\bar{J}\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \bar{J}\bar{Y}) ,$$

(1.8) $$\bar{\nabla}_{\bar{X}}\bar{J} = 0 ,$$

where $\bar{\nabla}$ denotes the operator of covariant differentiation with respect to the Riemannian connection of $\bar{g}$.

Since the curvature tensor of S^n is of the form

$$R'(X', Y')Z' = g'(Y', Z')X' - g'(X', Z')Y' ,$$

the curvature tensor of $S^n \times S^n$ is given by [5], [6]

(1.9) $$\begin{aligned} &\bar{R}(\bar{X}, \bar{Y})\bar{Z} \\ &= \tfrac{1}{2}\{\bar{g}(\bar{Y}, \bar{Z})\bar{X} - \bar{g}(\bar{X}, \bar{Z})\bar{Y} + \bar{g}(\bar{J}\bar{Y}, \bar{Z})\bar{J}\bar{X} - \bar{g}(\bar{J}\bar{X}, \bar{Z})\bar{J}\bar{Y}\} , \end{aligned}$$

from which we can easily see that $S^n \times S^n$ is an Einstein manifold because of (1.6) and (1.7).

Now, let M be a hypersurface of $S^n \times S^n$, and B the differential of the imbedding i of M into $S^n \times S^n$. Let X be a tangent vector field of M. Applying $\bar{J}$ to BX and to the unit normal vector N of M, we obtain vector fields $\bar{J}BX$ and $\bar{J}N$ which can be written in the following way:

(1.10) $$\bar{J}BX = BfX + u(X)N ,$$

(1.11) $$\bar{J}N = BU + \lambda N .$$

Then f, u, U and λ define a symmetric linear transformation of the tangent bundle of M, a 1-form, a vector field and a function on M respectively. Moreover, we easily see that

$$g(U, X) = u(X) ,$$

where g is the induced Riemannian metric on M.

If u is identically 0, then M is said to be an invariant hypersurface, that is, the tangent space $T_x(M)$ is invariant under $\bar{J}$. We will see later (1.20) that this is equivalent to $\lambda^2 = 1$.

We denote by ∇ the operator of covariant differentiation with respect to the Riemannian connection of g. Then the Gauss and Weingarten equations are given by

$$\bar{\nabla}_{BX}BY = B\nabla_X Y + h(X, Y)N , \tag{1.12}$$

$$\bar{\nabla}_{BX}N = -BHX , \tag{1.13}$$

where h is the second fundamental tensor of the hypersurface and satisfies

$$h(X, Y) = g(HX, Y) = g(X, HY) = h(Y, X) .$$

The relation between the curvature tensors of $S^n \times S^n$ and of M is given by

$$\begin{aligned} \bar{R}(BX, BY)BZ = B\{R(X, Y)Z - h(Y, Z)HX + h(X, Z)HY\} \\ + \{\nabla_X h(Y, Z) - \nabla_Y h(X, Z)\}N . \end{aligned} \tag{1.14}$$

Substituting (1.9) into (1.14) and making use of (1.10), we obtain

$$\begin{aligned} R(X, Y)Z = \tfrac{1}{2}\{g(Y, Z)X - g(X, Z)Y + g(fY, Z)fX - g(fX, Z)fY\} \\ + h(Y, Z)HX - h(X, Z)HY , \end{aligned} \tag{1.15}$$

$$(\nabla_X H)Y - (\nabla_Y H)X = \tfrac{1}{2}(u(X)fY - u(Y)fX) . \tag{1.16}$$

We apply $\bar{J}$ to both sides of (1.10). Then by virtue of (1.10) and (1.11) we get

$$BX = B(f^2X + u(X)U) + (u(fX) + \lambda u(X))N ,$$

which implies that

$$f^2X = X - u(X)U , \tag{1.17}$$

$$u(fX) = -\lambda u(X) . \tag{1.18}$$

Applying $\bar{J}$ to both sides of (1.11), we obtain

$$N = B(fU + \lambda U) + (u(U) + \lambda^2)N ,$$

that is,

$$fU = -\lambda U , \tag{1.19}$$

$$u(U) = g(U, U) = 1 - \lambda^2 . \tag{1.20}$$

Pick an orthonormal frame $\bar{E}_\alpha$, $\alpha = 1, \cdots, 2n$ in such a way that the first $2n-1$ $\bar{E}_\alpha$'s satisfy $\bar{E}_i = BE_i$, and $\bar{E}_{2n} = N$. Then because of (1.6) and (1.10) we have

$$
\begin{aligned}
\operatorname{tr} f &= \sum_{i=1}^{2n-1} g(fE_i, E_i) = \sum_{i=1}^{2n-1} \bar{g}(BfE_i, BE_i) = \sum_{i=1}^{2n-1} \bar{g}(\bar{J}BE_i, BE_i) \\
&= \sum_{\alpha=1}^{2n} (\bar{J}\bar{E}_\alpha, \bar{E}_\alpha) - \bar{g}(\bar{J}N, N) = \operatorname{tr} \bar{J} - \lambda = -\lambda\ .
\end{aligned}
\tag{1.21}
$$

Differentiating (1.10) convariantly and making use of (1.10), (1.11), (1.12) and (1.13), we have

$$
\begin{aligned}
&\bar{J}(B\nabla_Y X + h(X, Y)N) \\
&\quad = B\nabla_Y(fX) + h(fX, Y)N + (\nabla_Y u)(X)N + u(\nabla_Y X)N - u(X)BHY\ ,
\end{aligned}
$$

from which we have

$$(\nabla_Y f)X = h(X, Y)U + u(X)HY\ , \tag{1.22}$$

$$(\nabla_Y u)(X) = \lambda h(X, Y) - h(fX, Y)\ . \tag{1.23}$$

Similarly differentiating (1.11) covariantly, we get

$$\nabla_X U = -fHX + \lambda HX\ , \tag{1.24}$$

$$X\lambda = -2h(U, X) = -2u(HX)\ . \tag{1.25}$$

We also have

$$\operatorname{tr} \nabla_X H = \nabla_X \operatorname{tr} H = \sum_i g((\nabla_{E_i} H)X, E_i)\ , \tag{1.26}$$

where $E_i, i = 1, \cdots, 2n-1$ are the vector fields which extend to an orthonormal basis in $T_x(M)$ in a neighborhood of x.

2. Integral formulas for the hypersurface

Consider the function $S = \operatorname{tr} H^2$. Since the unit normal vector N is defined up to a sign, S is defined globally on M. We will now compute the Laplacian ΔS. We have

$$
\begin{aligned}
XS &= \nabla_X S = \nabla_X \operatorname{tr} H^2 = \operatorname{tr} \nabla_X H^2 \\
&= \operatorname{tr} (\nabla_X H)H + \operatorname{tr} H(\nabla_X H) = 2 \operatorname{tr} (\nabla_X H)H\ ,
\end{aligned}
$$

from which we have

$$YXS = 2\,\mathrm{tr}\,(\nabla_Y(\nabla_X H))H + 2\,\mathrm{tr}\,(\nabla_X H)(\nabla_Y H)\ ,$$
$$(\nabla_Y X)S = 2\,\mathrm{tr}\,(\nabla_{\nabla_Y X} H)H\ .$$

Hence

$$\tag{2.1} \tfrac{1}{2}\Delta S = \sum_{i=1}^{2n-1} \{\mathrm{tr}\,((\nabla_{E_i}\nabla_{E_i} H - \nabla_{\nabla_{E_i}E_i} H)H) + \mathrm{tr}\,(\nabla_{E_i} H)^2\}\ .$$

Putting

$$K(X, Y) = \nabla_Y(\nabla_X H) - \nabla_{\nabla_Y X} H\ ,$$

we have

$$\tag{2.2} K(X, Y)Z = K(Y, X)Z + R(X, Y)(HZ) - H(R(X, Y)Z)\ .$$

Let $E_i, i = 1, \cdots, 2n-1$ be an orthonormal basis in $T_x(M)$, and extend the E_i to vector fields in a neighborhood of x in such a way that $\nabla_Y E_i = 0$ at x. Let X be a vector field such that $\nabla_Y X = 0$ at x. Replacing X, Y, and Z in (2.2) by E_i, X and E_i respectively and taking account of (1.16) and the fact that $\nabla_Y E_i = 0$, $\nabla_Y X = 0$, we obtain

$$\begin{aligned} K(E_i, X)E_i &= (\nabla_{E_i}(\nabla_X H))E_i - (\nabla_{\nabla_{E_i}X} H)E_i \\ &= \nabla_{E_i}((\nabla_X H)E_i) - (\nabla_X H)(\nabla_{E_i} E_i) \\ &= \nabla_{E_i}\{(\nabla_{E_i} H)X + \tfrac{1}{2}(u(X)fE_i - u(E_i)fX)\}\ . \end{aligned}$$

Continuing this computation and making use of (1.22), (1.23), we have at x

$$\begin{aligned} K(E_i, X)E_i = (\nabla_{E_i}(\nabla_{E_i} H))X &+ \tfrac{1}{2}\{(\lambda h(X, E_i) - h(fX, E_i))fE_i \\ &+ u(x)(h(E_i, E_i)U + u(E_i)HE_i) - (\lambda h(E_i, E_i) \\ &- h(fE_i, E_i))fX - u(E_i)(h(E_i, X)U + u(X)HE_i)\}\ , \end{aligned}$$

from which we get

$$\begin{aligned} \sum_{i=1}^{2n-1} K(E_i, X)E_i = \sum_{i=1}^{2n-1} \{K(E_i, E_i)X &+ \tfrac{1}{2}(\lambda h(X, E_i) - h(fX, E_i))fE_i\} \\ &+ \tfrac{1}{2}\Big\{u(X)(\mathrm{tr}\,H)U + u(X)\sum_{i=1}^{2n-1} g(U, E_i)HE_i \\ &\quad - \lambda(\mathrm{tr}\,H)fX + (\mathrm{tr}\,Hf)fX \\ &\quad - \sum_{i=1}^{2n-1} g(U, E_i)h(E_i, X)U - \sum_{i=1}^{2n-1} u(E_i)u(X)HE_i\Big\}\ . \end{aligned}$$

Here

$$\sum_{i=1}^{2n-1} h(X, E_i)fE_i = f\left(\sum_{i=1}^{2n-1} g(HX, E_i)E_i\right) = fHX ,$$

$$\sum_{i=1}^{2n-1} h(fX, E_i)fE_i = fHfX ,$$

$$\sum_{i=1}^{2n-1} u(E_i)HE_i = \sum_{i=1}^{2n-1} g(U, E_i)HE_i = H\left(\sum_{i=1}^{2n-1} g(U, E_i)E_i\right) = HU ,$$

$$\sum_{i=1}^{2n-1} g(U, E_i)h(E_i, X) = \sum_{i=1}^{2n-1} g(U, E_i)g(HX, E_i)$$
$$= \sum_{i=1}^{2n-1} g(HX, g(U, E_i)E_i) = g(HX, U) .$$

Hence

$$\begin{aligned}\sum_{i=1}^{2n-1} K(E_i, X)E_i = \sum_{i=1}^{2n-1} K(E_i, E_i)X + \tfrac{1}{2}\{\lambda fHX - fHfX + u(x)(\operatorname{tr} H)U \\ + (\operatorname{tr} Hf)fX - \lambda(\operatorname{tr} H)fX - g(HX, U)U\} .\end{aligned} \tag{2.3}$$

Thus we get from (2.2) and (2.3) that

$$\begin{aligned}\sum_{i=1}^{2n-1} K(E_i, E_i)X + \tfrac{1}{2}\{\lambda fHX - fHfX + u(X)(\operatorname{tr} H)U + (\operatorname{tr} Hf)fX \\ - \lambda(\operatorname{tr} H)fX - g(HX, U)U\} \\ = \sum_{i=1}^{2n-1} \{K(X, E_i)E_i + R(E_i, X)(HE_i) - H(R(E_i, X)E_i)\} .\end{aligned}$$

We now assume that the hypersurface M has constant mean curvature, that is, $\operatorname{tr} H = \text{const}$. Then (1.26) and the choice of E_i and X show that

$$\sum_{i=1}^{2n-1} K(X, E_i)E_i = \sum_{i=1}^{2n-1} (\nabla_X(\nabla_{E_i}H) - \nabla_{\nabla_X E_i}H)E_i = \sum_{i=1}^{2n-1} (\nabla_X(\nabla_{E_i}H))E_i = 0 .$$

Hence we get

$$\begin{aligned}\sum_{i=1}^{2n-1} K(E_i, E_i)X = -\tfrac{1}{2}\{\lambda fHX - fHfX + u(X)(\operatorname{tr} H)U \\ + (\operatorname{tr} Hf)fX - \lambda(\operatorname{tr} H)fX - g(HX, U)U\} \\ + \sum_{i=1}^{2n-1} \{R(E_i, X)(HE_i) - H(R(E_i, X)E_i)\} .\end{aligned} \tag{2.4}$$

On the other hand, by (1.15) we have

$$\begin{aligned}\sum_{i=1}^{2n-1} R(E_i, X)(HE_i) = \tfrac{1}{4}\{g(X, HE_i)E_i - g(E_i, HE_i)X + g(fX, HE_i)fE_i \\ - g(fE_i, HE_i)fX\} + h(X, HE_i)HE_i - h(E_i, HE_i)HX\end{aligned}$$

$$= \tfrac{1}{2}\{HX - (\mathrm{tr}\, H)X + fHfX - (\mathrm{tr}\, Hf)fX\} + H^3X - (\mathrm{tr}\, H^2)HX ,$$

$$\sum_{i=1}^{2n-1} H(R(E_i, X)E_i) = \tfrac{1}{2}\{g(X, E_i)HE_i - g(E_i, E_i)HX + g(fX, E_i)HfE_i - g(fE_i, E_i)HFX\} + h(X, E_i)HE_i - h(E_i, E_i)HX$$
$$= \tfrac{1}{2}\{2(1-n)HX + Hf^2X - (\mathrm{tr}\, f)HfX\} + H^3X - (\mathrm{tr}\, H)H^2X .$$

Substituting the above two equations into (2.4) and making use of (1.17), we have

$$\sum_{i=1}^{2n-1} K(E_i, E_i)X = -\tfrac{1}{2}\{\lambda fHX - 2fHX - u(X)(\mathrm{tr}\, H)U + 2(\mathrm{tr}\, Hf)fX - \lambda(\mathrm{tr}\, H)fX - g(HX, U)U + (\mathrm{tr}\, H)X + 2(\mathrm{tr}\, H^2)HX - 2(n-1)HX - u(X)HU + \lambda HfX - 2(\mathrm{tr}\, H)H^2X\} ,$$

which implies that

$$2\sum_{i=1}^{2n-1} K(E_i, E_i)HX = -\lambda fH^2X + 2fHfHX + u(HX)(\mathrm{tr}\, H)U - 2(\mathrm{tr}\, Hf)fHX + \lambda(\mathrm{tr}\, H)fHX + g(HU, HX)U - (\mathrm{tr}\, H)HX - 2(\mathrm{tr}\, H^2)H^2X + 2(n-1)H^2X + u(HX)HU - \lambda HfHX + 2(\mathrm{tr}\, H)H^3X .$$

Thus we have

$$\begin{aligned} \Delta S &= 2\sum_{j,i=1}^{2n-1} \{g(K(E_i, E_i)HE_j, E_j) + \mathrm{tr}\,(\nabla_{E_i}H)^2\} \\ &= -2\lambda\, \mathrm{tr}\, fH^2 + 2\,\mathrm{tr}\,(fH)^2 + (\mathrm{tr}\, H)g(HU, U) - 2(\mathrm{tr}\, Hf)^2 \\ &\quad + \lambda(\mathrm{tr}\, H)\,\mathrm{tr}\, fH + 2g(HU, HU) - (\mathrm{tr}\, H)^2 \\ &\quad - 2S(S-(n-1)) + 2(\mathrm{tr}\, H)\,\mathrm{tr}\, H^3 + 2g(\nabla H, \nabla H) , \end{aligned} \tag{2.5}$$

where the metric g is extended to the tensor space in the standard fashion. In particular, if the hypersurface M is minimal, that is, if $\mathrm{tr}\, H = 0$, then

$$\begin{aligned} \tfrac{1}{2}\Delta S &= -\lambda\, \mathrm{tr}\, fH^2 + \mathrm{tr}\,(fH)^2 - (\mathrm{tr}\, Hf)^2 + g(HU, HU) \\ &\quad + S((n-1) - S) + g(\nabla H, \nabla H) . \end{aligned} \tag{2.6}$$

Next we want to compute $\mathrm{div}\,((\mathrm{tr}\, fH)U - fHU)$. Since $\mathrm{div}\, Z = \sum_{i=1}^{2n-1} g(\nabla_{E_i}Z, E_i)$ for any vector field Z, we first have

$$\begin{aligned}\nabla_X(\operatorname{tr}(fH)U) &= (\nabla_X(\operatorname{tr} fH))U + (\operatorname{tr} fH)\nabla_X U \\ &= \sum_{i=1}^{2n-1} \nabla_X(g(fHE_i, E_i))U - (\operatorname{tr} fH)fHX + \lambda(\operatorname{tr} fH)HX ,\end{aligned} \tag{2.7}$$

because of (1.24). Remembering the choice of E_i and (1.22), we have at x

$$\begin{aligned}&\nabla_X g(fHE_i, E_i) \\ &\quad = g((\nabla_X f)HE_i, E_i) + g(f(\nabla_X H)E_i, E_i) \\ &\quad = g(g(H^2E_i, X)U + u(HE_i)HX, E_i) + g(f(\nabla_X H)E_i, E_i) \\ &\quad = g(H^2E_i, X)g(U, E_i) + g(U, HE_i)g(HX, E_i) + g(f(\nabla_X H)E_i, E_i) \\ &\quad = g(H^2X, E_i)g(U, E_i) + g(HU, E_i)g(HX, E_i) + g(f(\nabla_X H)E_i, E_i) .\end{aligned}$$

Therefore

$$\sum_{i=1}^{2n-1} \nabla_X g(fHE_i, E_i) = 2g(H^2X, U) + \operatorname{tr} f(\nabla_X H) .$$

Substituting this into (2.7), we have

$$\nabla_X(\operatorname{tr}(fH)U) = 2g(H^2X, U)U + (\operatorname{tr} f\nabla_X H)U - (\operatorname{tr} fH)fHX + \lambda(\operatorname{tr} fH)HX ,$$

from which it follows that

$$\begin{aligned}\operatorname{div}(\operatorname{tr}(fH)U) &= \sum_{i=1}^{2n-1}\{2g(H^2E_i, U)g(U, E_i) + (\operatorname{tr} f\nabla_{E_i}H)g(E_i, U)\} \\ &\quad - (\operatorname{tr} fH)^2 + \lambda(\operatorname{tr} fH)\operatorname{tr} H .\end{aligned}$$

Here

$$\begin{aligned}g(H^2E_i, U)g(U, E_i) &= g(E_i, H^2U)g(U, E_i) = g(H^2U, U) = g(HU, HU) , \\ (\operatorname{tr} f\nabla_{E_i}H)g(E_i, U) &= (\operatorname{tr} f\nabla_{g(E_i,U)E_i}H) = \operatorname{tr} f\nabla_U H .\end{aligned}$$

Hence

$$\begin{aligned}\operatorname{div}((\operatorname{tr}(fH))U) &= 2g(HU, HU) + \operatorname{tr}(f\nabla_U H) - (\operatorname{tr} fH)^2 \\ &\quad + \lambda(\operatorname{tr} fH)\operatorname{tr} H .\end{aligned} \tag{2.8}$$

On the other hand we have, from (1.22), (1.24) and (1.16),

$$\begin{aligned}\nabla_X(fHU) &= (\nabla_X f)HU + f(\nabla_X H)U + fH\nabla_X U \\ &= g(H^2U, X)U + g(HU, U)HX + f((\nabla_U H)X \\ &\qquad - \tfrac{1}{2}u(X)fU + \tfrac{1}{2}u(U)fX) + fH(-fHX + \lambda HX) \\ &= g(H^2U, X)U + g(HU, U)HX + f(\nabla_U H)X - \tfrac{1}{2}\lambda^2 u(X)U\end{aligned}$$

$$+ \tfrac{1}{2}(1 - \lambda^2)(X - u(X)U) - (fH)^2X + \lambda fH^2X ,$$

from which it follows that

$$\begin{aligned} \operatorname{div}(fHU) = {} & g(HU, HU) + g(HU, U)(\operatorname{tr} H) + \operatorname{tr} f\nabla_U H \\ & + (n - 1)(1 - \lambda^2) - \operatorname{tr}(fH)^2 + \lambda \operatorname{tr} fH^2 . \end{aligned} \tag{2.9}$$

Subtracting (2.9) from (2.8), we get

$$\begin{aligned} \operatorname{div}((\operatorname{tr} fH)U - fHU) = {} & g(HU, HU) - (\operatorname{tr} fH)^2 + \lambda(\operatorname{tr} fH) \operatorname{tr} H \\ & - (\operatorname{tr} H)g(HU, U) + \operatorname{tr}(fH)^2 - \lambda \operatorname{tr} fH^2 \\ & + (n - 1)(1 - \lambda^2) . \end{aligned} \tag{2.10}$$

In particular, if M is minimal, we get

$$\begin{aligned} & \operatorname{div}((\operatorname{tr} fH)U - fHU) \\ & = g(HU, HU) - (\operatorname{tr} fH)^2 + \operatorname{tr}(fH)^2 - \lambda \operatorname{tr} fH^2 + (n - 1)(1 - \lambda^2) . \end{aligned} \tag{2.11}$$

Now we compute div ((tr $H(U)$. Since M has constant mean curvature, we have

$$\nabla_X((\operatorname{tr} H)U) = (\operatorname{tr} H)\nabla_X U = (\operatorname{tr} H)(-fHX + \lambda HX) ,$$

which implies that

$$\operatorname{div}((\operatorname{tr} H)U) = -(\operatorname{tr} H) \operatorname{tr} fH + \lambda(\operatorname{tr} H)^2 . \tag{2.12}$$

Thus we have

$$\begin{aligned} & \tfrac{1}{2}\Delta S - \operatorname{div}((\operatorname{tr} fH)U - fHU) - \tfrac{1}{2} \operatorname{div}((\operatorname{tr} H)U) \\ & \quad = \tfrac{3}{2}(\operatorname{tr} H)g(HU, U) - \tfrac{1}{2}(\lambda - 1)(\operatorname{tr} H) \operatorname{tr} fH - \tfrac{1}{2}(1 + \lambda)(\operatorname{tr} H)^2 \\ & \quad\quad - S(S - (n - 1)) + (\operatorname{tr} H) \operatorname{tr} H^3 - (n - 1)(1 - \lambda^2) + g(\nabla H, \nabla H) . \end{aligned}$$

Assume that the hypersurface M is compact and orientable. Integrating the above equation over M, we get, because of Green-Stokes' theorem,

$$\begin{aligned} \int_M & \{\tfrac{3}{2}(\operatorname{tr} H)g(HU, U) - \tfrac{1}{2}(\lambda - 1)(\operatorname{tr} H) \operatorname{tr} fH \\ & - \tfrac{1}{2}(1 + \lambda)(\operatorname{tr} H)^2 - S(S - (n - 1)) + (\operatorname{tr} H) \operatorname{tr} H^3 \\ & - (n - 1)(1 - \lambda^2) + g(\nabla H, \nabla H)\}dM = 0 . \end{aligned} \tag{2.13}$$

In particular, if the hypersurface is minimal, then

$$\int_M \{S(n - 1) - S) - (n - 1)(1 - \lambda^2) + g(\nabla H, \nabla H)\}dM = 0 . \tag{2.14}$$

Similarly, if we integrate

$$\tfrac{1}{2}\Delta S - \operatorname{div}((\operatorname{tr} fH)U - fHU) + \operatorname{div}((\operatorname{tr} H)U) ,$$

then we have

$$\begin{aligned}\int_M \{&\tfrac{3}{2}(\operatorname{tr} H)g(HU, U) - \tfrac{1}{2}(\lambda + 1)(\operatorname{tr} H) \operatorname{tr} fH \\ &- \tfrac{1}{2}(1 - \lambda)(\operatorname{tr} H)^2 - S(S - (n - 1)) + (\operatorname{tr} H) \operatorname{tr} H^3 \\ &- (n - 1)(1 - \lambda^2) + g(\nabla H, \nabla H)\}dM = 0 .\end{aligned} \tag{2.15}$$

From (2.14) we get easily

Theorem 2.1. *A compact orientable minimal hypersurface of $S^n \times S^n$ $(n > 1)$ satisfying*

$$\int_M (S^2 - (n - 1)S)dM \geq \int_M \|\nabla H\|^2 \, dM \tag{2.16}$$

is an invariant hypersurface.

Corollary 2.2. *A compact orientable minimal hypersurface with parallel second fundamental tensor of $S^n \times S^n$ satisfying $S \geq n - 1$ is an invariant hypersurface.*

Corollary 2.3. *A compact orientable totally geodesic hypersurface of $S^n \times S^n$ is an invariant hypersurface.*

3. Invariant hypersurfaces of $S^n \times S^n$

In this section we assume that the hypersurface M is invariant, i.e., (1.10) can be wrttten as

$$\bar{J}BX = BfX . \tag{3.1}$$

Since the 1-form u and the vector field U vanish identically, we have

$$f^2X = X , \tag{3.2}$$

$$1 - \lambda^2 = 0 , \tag{3.3}$$

$$\nabla_X f = 0 , \tag{3.4}$$

$$X\lambda = 0 . \tag{3.5}$$

We may assume that[1] $\lambda = 1$ in the following discussions. Then the formulas (2.13) and (2.14) become

[1] If we take $\lambda = -1$, then we use (2.15) instead of (2.13) and get the same results.

$$(3.6)\qquad \int_M \{S((n-1)-S)-(\operatorname{tr} H)^2+(\operatorname{tr} H)\operatorname{tr} H^3+g(\nabla H,\nabla H)\}dM=0\ ,$$

$$(3.7)\qquad \int_M \{S((n-1)-S)+g(\nabla H,\nabla H)\}dM=0\ ,$$

respectively. Thus we get

Theorem 3.1. *Let M be a compact orientable invariant minimal hypersurface of $S^n \times S^n$. Then either M is the totally geodesic hypersurface or $S \equiv n-1$, or $S(x) > n-1$ at some $x \in M$.*

Corollary 3.2. *Let M be a compact orientable invariant minimal hypersurface of $S^n \times S^n$. If $S < n-1$, then M is a totally geodesic hypersurface.*

Now let

$$T_1(x)=\{X \in T_x(M)\,;\, fX=X\}\ ,\qquad T_{-1}(x)=\{X\in T_x(M)\,;\, fX=-X\}\ .$$

Then the correspondence of $x \in M$ to $T_1(x)$ and that to $T_{-1}(x)$ define $(n-1)$-dimensional and n-dimensional distributions respectively, since $\operatorname{tr} f=-\lambda=-1$. By virtue of (3.4) it follows that both distributions are involutive. We easily see that if $X \in T_1(x)$ and $Y \in T_{-1}(x)$, then $\nabla_Y X \in T_1(X)$ and $\nabla_X Y \in T_{-1}(x)$. Hence both distributions are parallel. Moreover, for the vector fields X and Y chosen in the above way, we have $g(\nabla_Z X, Y)=0$ and $g(\nabla_W Y, X)=0$, where $Z \in T_1(x)$ and $W \in T_{-1}(X)$. Thus the integral manifolds of $T_1(X)$ and $T_{-1}(X)$ are both totally geodesic in M. By standard arguments (see [2]) we know that M is a product of the integral manifolds of the distributions $T_1(x)$ and $T_{-1}(x)$. In the next step we want to show that the integral submanifold of $T_{-1}(x)$ is S^n.

Let $X \in T_{-1}(X)$. Then by virtue of (1.1), (1.4) it follows that

$$\bar{P}BX=\tfrac{1}{2}(IBX+\bar{J}BX)=\tfrac{1}{2}(BX+BfX)=0\ .$$

Thus BX belongs to the tangent space $T(S^n)$ which is defined by $V_Q=\{\bar{X}\,;\ \bar{Q}\bar{X}=\bar{X}\}$. Conversely, if we take a vector field $\bar{X}$ belonging to V_Q, $\bar{X}$ can be written as a sum of the tangential components and the normal components. So we put

$$\bar{X}=BX+\alpha N\ .$$

Applying $\bar{P}$ to the above equation, we have

$$\begin{aligned}0=\bar{P}\bar{X}=\bar{P}BX+\alpha\bar{P}N&=\tfrac{1}{2}\{(IBX+\bar{J}BX)+\alpha(IN+\bar{J}N)\}\\&=\tfrac{1}{2}\{BX+BfX+2\alpha N\}\ ,\end{aligned}$$

from which we have

$$fX=-X\ ,\qquad \alpha=0\ .$$

This means that $\bar{X} = BX$, and consequently V_Q is isomorphic to $BT_{-1}(x)$. Thus, the integral submanifold being unique since M is complete, the integral submanifold of $T_{-1}(x)$ must be S^n. If $X \in T_1(x)$, then the same discussion as above shows that $BX \in V_P = \{\bar{X}; \bar{P}\bar{X} = \bar{X}\}$. Since the integral submanifold of V_P is another S^n, the integral submanifold of $T_1(X)$ is a hypersurface of S^n. Thus we have

Theorem 3.3. *A complete invariant hypersurface of $S^n \times S^n$ is a product manifold $M' \times S^n$, where M' is a hypersurface of S^n.*

In order to get further results, we prove

Lemma 3.4. *Let P and Q be the projection of $T(M)$ into $T(M')$ and $T(S^n)$ respectively. Then we have*

$$HQX = 0 \,. \tag{3.8}$$

Proof. By the definitions of $\bar{J}, P, Q$, we have

$$\bar{J}BQX = (\bar{P} - \bar{Q})BQX = B(\bar{P} - \bar{Q})\bar{Q}BX = -\bar{Q}BX = -BQX \,,$$

since $V_Q = BT_{-1}(x)$. Hence

$$\bar{\nabla}_{BY}(\bar{J}BQX) = -\bar{\nabla}_{BY}(BQX) = -B\nabla_Y(QX) - h(Y, QX)N \,. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned} \bar{\nabla}_{BY}(\bar{J}BQX) &= \bar{J}(B\nabla_Y(QX) + h(Y, QX)N) \\ &= -B\nabla_Y(QX) + h(Y, QX)\bar{J}N \\ &= -B\nabla_Y(QX) + h(Y, QX)N \,, \end{aligned} \tag{3.10}$$

because of the fact that $\nabla_Y(QX) \in V_Q$ and $\bar{J}N = N$.

Comparing (3.9) and (3.10), we have $h(Y, QX) = 0$, from which (3.8) follows.

We consider the immersion i': $M' \to M' \times S^n = M$, and denote the differential of i' by B'. Then we have

$$\bar{\nabla}_{BB'Y'}BB'X' = BB'\nabla'_{Y'}X' + \sum_{A=1}^{n+1} h'_A(X', Y')N'_A \,, \tag{3.11}$$

where $X', Y' \in T(M')$, and h'_A's are the second fundamental tensor with respect to the normals N'_A. Now we choose the last normal N'_{n+1} in such a way that N'_{n+1} is the unit normal to M' in S^n.

On the other hand, we have

$$\bar{\nabla}_{BB'Y'}BB'X' = B\nabla_{B'Y'}B'X' + h(B'X', B'Y')N \,,$$

from which it follows that

$$(3.12)\quad \bar{\nabla}_{BB'Y'}BB'X' = BB'\nabla'_{Y'}X' + \sum_{\alpha=1}^{n} h_\alpha(X',Y')BN_\alpha + h(B'X',B'Y')N\ .$$

Comparing (3.11) and (3.12) and remembering the choice of normals, we get

$$(3.13)\quad \begin{aligned} h_\alpha(X',Y') &= h'_\alpha(X',Y') \qquad \text{for } \alpha = 1,\cdots,n\ ,\\ h(B'X',B'Y') &= h'_{n+1}(X',Y')\ . \end{aligned}$$

Since M' is a totally geodesic submanifold in $M' \times S^n$, it follows that $h_\alpha(X',Y') = 0$ for $\alpha = 1,\cdots,n$. Thus

$$(3.14)\quad \sum_{A=1}^{n+1} \operatorname{tr} H_A'^P = \operatorname{tr} {H'_{n+1}}^P\ ,$$

for any natural number P. Furthermore,

$$\operatorname{tr} H^P = \sum_{i=1}^{2n-1} g(H^P E_i, E_i) = \sum_{A=1}^{n-1} g(H^P B'E_A, B'E_A) + \sum_{t=1}^{n} g(H^P N'_t, N'_t)\ ,$$

where N'_t, $t = 1,\cdots,n$ are unit normals to M' in $M' \times S^n$. Since there exist X'_t in $T(M)$ such that $N'_t = QX_t$, we have $H^P N'_t = 0$ because of Lemma 3.2. Thus we get

$$\operatorname{tr} H^P = \sum_{A=1}^{n-1} g(H^P B'E_A, B'E_A) = \sum_{A=1}^{n-1} g({H'_{n+1}}^P E_A, E_A) = \operatorname{tr} {H'_{n+1}}^P\ .$$

This shows that, once we fix a choice of normals in the above way, $\operatorname{tr} H^P$ is a function on M'. The immersion $i\colon M \to S^n \times S^n$ being $i' \times \mathrm{id}\colon M' \times S^n \to S^n \times S^n$, we have that the second fundamental tensor H'_{n+1} is identical with that of M' in S^n. Thus, denoting the second fundamental tensor of M' in S^n by H' and using (3.6), (3.7) and Fubini theorem of measure theory, we have that

$$(3.15)\quad \begin{aligned} &\left(\int_{M'} \{S'((n-1)-S') - (\operatorname{tr} H')^2 + (\operatorname{tr} H')\operatorname{tr} H'^3\}dM'\right)\operatorname{vol} S^n\\ &\qquad + \int_M g(\nabla H, \nabla H)dM = 0\ , \end{aligned}$$

$$(3.16)\quad \left(\int_{M'} S'((n-1)-S')dM'\right)\operatorname{vol} S^n + \int_M g(\nabla H,\nabla H)dM = 0\ ,$$

where $S' = \operatorname{tr} H'^2 = \operatorname{tr} H^2 = S$.

We first consider the case where M is a minimal hypersurface. In this case, if $S = 0$, it follows that $S' = 0$ and consequently M' is the totally geodesic great sphere of S^n. Thus we have $M = S^{n-1} \times S^n$, where both S^{n-1} and S^n are of radius 1.

If $S = n - 1$, then $S' = n - 1$. Applying Chern-do Carmo-Kobayashi's theorem, we have $M' = S^m(\sqrt{m/(n-1)}) \times S^{n-m-1}(\sqrt{(n-m-1)/(n-1)})$, where we denote the radius of spheres in the parentheses. Hence we have $M = S^m(\sqrt{m/(n-1)}) \times S^{n-m-1}(\sqrt{(n-m-1)/(n-1)}) \times S^n(1)$.

Theorem 3.5. *The* $S^m(\sqrt{m/(n-1)}) \times S^{n-m-1}(\sqrt{(n-m-1)/(n-1)}) \times S^n(1)$ *in* $S^n \times S^n$ *are the only compact orientable invariant minimal hypersurfaces of* $S^n \times S^n$ *satisfying* $S = n - 1$.

Combining Theorem 3.1 and Theorem 3.5, we have

Theorem 3.6. *The* $S^{n-1}(1) \times S^n(1)$ *and*

$$S^m(\sqrt{m/(n-1)}) \times S^{n-m-1}(\sqrt{(n-m-1)/(n-1)}) \times S^n(1)$$

are the only compact orientable invariant minimal hypersurfaces of $S^n \times S^n$ *satisfying* $S \leq n - 1$.

Next we consider the formula (3.15). We assume that M has principal curvatures $\lambda_1, \cdots, \lambda_{2n-1}$ such that for any pair of $\lambda_i, \lambda_j, i, j = 1, \cdots, 2n-1$, $\lambda_i \lambda_j \geq 0$ holds, that is, M has principal curvatures of the same sign or 0. Then by means of the Cauchy-Schwarz inequality, we have

$$(\operatorname{tr} H) \operatorname{tr} H^3 - S^2 = \sum_{i=1}^{2n-1} (\lambda_i^{1/2})^2 \sum_{i=1}^{2n-1} (\lambda_i^{3/2})^2 - \sum_{i=1}^{2n-1} \lambda_i^{1/2} \lambda_i^{3/2} \geq 0 .$$

Thus (3.6) becomes

$$\int_M \{(n-1) \operatorname{tr} H^2 - (\operatorname{tr} H)^2 + g(\nabla H, \nabla H)\} dM \leq 0 ,$$

which, together with (3.15), implies

$$\begin{aligned}
&\left(\int_{M'} \left\{(n-1)\left(\operatorname{tr} H'^2 - \frac{1}{n-1}(\operatorname{tr} H')^2\right\} dM'\right) \operatorname{vol} S^n \\
&= (n-1)\left(\int_{M'} \operatorname{tr}\left(H' - \frac{1}{n-1}(\operatorname{tr} H')I\right)^2 dM'\right) \operatorname{vol} S^n \\
&= (n-1)\left(\int_{M'} \operatorname{tr}\left\{\left(H' - \frac{1}{n-1}(\operatorname{tr} H')I\right)^t\right.\right. \\
&\qquad \left.\left. \cdot \left(H' - \frac{1}{n-1}(\operatorname{tr} H')I\right)\right\} dM'\right) \operatorname{vol} S^n \\
&= (n-1)\left(\int_{M'} \left\| H' - \frac{1}{n-1}(\operatorname{tr} H')I \right\|^2 dM'\right) \operatorname{vol} S^n \leq 0 ,
\end{aligned}$$

which implies that

$$H' = \frac{1}{n-1}(\operatorname{tr} H')I \ .$$

This shows that M' is a totally umbilical hypersurface of S^n and consequently the small sphere of S^n. Thus we get

Theorem 3.7. *$S^{n-1}(r) \times S^n(1)$ is the only compact orientable invariant hypersurface of $S^n \times S^n$ with constant mean curvature, which has principal curvatures of the same sign or* 0.

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